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# Quantum group $\sigma$ models

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**Abstract.** Field-theoretic models for fields taking values in quantum groups are investigated. First we consider $SU_q(2)$ $\sigma$ model ($q$ real) expressed in terms of basic notions of non-commutative differential geometry. We discuss the case in which the $\sigma$ models fields are represented as products of conventional $\sigma$ fields and of the coordinate-independent algebra. An explicit example is provided by the $U_q(2)$ $\sigma$ model with $q^N = 1$, in which case quantum matrices $U_q(2)$ are realised as $2N \times 2N$ unitary matrices. Open problems are pointed out.

## 1. Introduction

The appearance of non-commutative entries in some matrices describing quantum inverse scattering methods for spin systems (see, e.g. [1–3]) has led to the introduction of the concept of a quantum matrix group [4–6]. From the algebraic point of view the description of a quantum group as a quasitriangular Hopf algebra was first given by Drinfeld [7], with the basic object being the non-commutative algebra of functions on a quantum group. The quantum extensions of all classical matrix groups (Cartan $A_n$, $B_n$, $C_n$ and $D_n$ series), describing the generators of Drinfeld's quantum algebra have also been given [4].

Moreover, the quantum counterparts of the homogeneous coset spaces (e.g.: spheres, $S^n_q$; projective spaces, $CP_q(n)$; etc) have also been found [4, 8–10].

In this paper we will consider fields taking values in quantum groups and we will discuss the corresponding $\sigma$ models. Let us recall that the usual $\sigma$-field $\phi(x)$ describes the mappings from the coordinate manifold $S$ into the target space $M$ (see e.g. [11]). In principle we can '$q$-deform' the target space $M$ ($M \to M_q$) as well as the coordinate manifold $S$ ($S \to S_q$), i.e. we can introduce three kinds of $\sigma$ models:

(a) with quantum deformation of the target manifold

$$\phi^q_a(x): \qquad x \in S \;\to\; \phi^q_a \in M_q \tag{1.1}$$

where the index $a$ ennumerates the local coordinates on $M_q$.

(b) with quantum deformation of the base manifold

$$\phi_a(x_q): \qquad x_q \in S_q \;\to\; \phi_a \in M. \tag{1.2}$$

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(c) with both manifolds deformed

$$\phi^q_a(x_q): \qquad x_q \in S_q \;\rightarrow\; \phi^q_a \in M_q. \tag{1.3}$$

Let us add that, analogously, there exist three types of supersymmetric models, corresponding to the three types of mappings, and that we believe that in the near future, the quantum group $\sigma$ models of all three types will be studied. In the present paper we will discuss mainly the case (1.1).

The description of quantum group $\sigma$ models can be presented in two different ways:

(1) We can consider quantum group $\sigma$ fields as fields satisfying at a point $x \in \mathrm{S}$ the quantum algebra, and so we can study the general properties which follow from the basic formulae of non-commutative differential geometry on quantum groups. Such an approach, which we shall call algebraic, was recently used by Arefeva and Volovich [12–13].

In the algebraic formulation of a quantum group $\sigma$ model one can repeat the major part of the geometric formulation of the standard approach to the $\sigma$ models (Cartan forms, Cartan structure relations, algebra of the covariant derivatives etc), provided that the exchange relations between quantum group valued $\sigma$ fields and their derivatives are properly introduced. For the $\mathrm{SU}_q(2)$ case the formulae of non-commutative geometry are well known (see e.g. [5, 14]) and so in the next section we shall present the algebraic formulation of the $\mathrm{SU}_q(2)$ $\sigma$ model. It appears that the effective use of the algebraic formulation requires several new developments, e.g.

(i) One should know the relations between quantum group $\sigma$ fields and their variations. Only when these are known one can derive the field equations from the action.

(ii) Any physical interpretation of the algebraic $\sigma$ model requires a construction giving real numbers. At present such a construction is not known; in particular, no proper distinction can be made between a classical and a quantum theory.

(2) In the main part of the paper we shall assume that the quantum group $\mathrm{G}_q$ $\sigma$ fields $\phi^q_A(x)$ are products of 'ordinary' functions $f_A(x) \in H$ and of the $x$ independent algebra $A$ related to the quantum group algebra. This approach gives us expressions which belong to the tensor product $H_G \otimes T_A$, where for an $n \times n$ matrix quantum group $\sigma$ model the first part $H_G$ is parametrised by an $n \times n$ matrix of 'classical' fields (suitably constrained standard $\mathrm{GL}(n)$ $\sigma$ fields), and $T_A$ carries the realization of the algebra $A$. If $A = f(G_q)$, the natural realization on the polynomial basis of the functions on the quantum group is infinite dimensional, and for $q$ real it can not be reduced to a finite dimensional case. In the second part of section 2 we will present our discussion of the $\mathrm{SU}_q(2)$ quantum group $\sigma$ model for the solutions satisfying the separability condition described above.

The realization $T_A$ can be described by finite matrices in one case: when $q$ is complex and $q^N = 1$ ($q = \mathrm{e}^{\mathrm{i}2\pi/N}$). It is worth mentioning that some realizations of the quantum groups for $q$ being the $N$th root of unity have recently been found to have physically relevant applications (see e.g. [15–17]). It appears, however, that the quantum deformations $\mathrm{O}_q(N)$ and $\mathrm{SU}_q(N)$ of the semi-simple groups $\mathrm{O}(N)$ and $\mathrm{SU}(N)$ which are the natural candidates for quantum group $\sigma$ fields, do not permit complex $q$ [4].

The simplest example of a compact quantum group with $q^N = 1$, the quantum group $\mathrm{U}_q(2)$, will be considered in section 3. There, we will first show that the group

$U_q(2)$ for complex values of $q$ can be obtained as a special case of a two-parameter deformation $GL_{q,p}(2, C)$ of the $2 \times 2$ general linear group. Then, using the results on the matrix realizations of $GL_q(N)$ for $q^N = 1$ [18, 19] we will embed the $U_q(2)$ $\sigma$ model with $q^N = 1$ ('anyonic $\sigma$ model') into the conventional $U(2N)$ $\sigma$ model. It appears that when we use such a representation we describe solutions that, for $D = 1$ (see equation (3.2)), satisfy the assumptions made by Arefeva and Volovich in their discussion of quantum group sigma models [12]. Finally, some open problems are discussed in section 4.

## 2. $SU_q(2)$ $\sigma$ model

### 2.1. *Algebraic formulation*

Let us first introduce the quantum group $SL_q(2, C)$ as the following Hopf bialgebra ($q$ complex) [4]:

(a) Multiplication:

$$\begin{aligned} &U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &ab = qba \qquad ac = qca \qquad cd = qdc \\ &bc = cb \qquad bd = qdb \\ &ad - qbc = da - q^{-1}cb = 1. \end{aligned} \tag{2.1}$$

(b) Coproduct

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.2}$$

(c) Inverse (antipode) and co-unit

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \qquad \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3}$$

For $q$ real we can introduce the following unitarity condition. For

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.4}$$

we put

$$U^\dagger = \begin{pmatrix} a^\star & b^\star \\ c^\star & d^\star \end{pmatrix}^T = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} = S(U) \tag{2.5}$$

which defines the $SU_q(2)$ quantum group, the matrix elements of which have the form

$$U = \begin{pmatrix} a & -qc^\star \\ c & a^\star \end{pmatrix}. \tag{2.6}$$

In this case the relations (2.1) take the form

$$\begin{aligned} ac &= qca \qquad ac^\star = qc^\star a \qquad cc^\star = c^\star c \\ aa^\star + q^2 cc^\star &= 1 \qquad a^\star a + c^\star c = 1. \end{aligned} \tag{2.7}$$

In order to define a quantum group $\sigma$ model we introduce the Cartan one-forms on $\mathrm{SU}_q(2)$

$$\Omega = U^\dagger \mathrm{d}U \leftrightarrow \Omega_{ik} = U_{ij}\,\mathrm{d}U_{jk}. \tag{2.8}$$

The formula (2.8) describes the left-invariant one-forms ($\Omega = \Omega_l$). The right-invariant forms are given in terms of the left-invariant forms in the same way as in the $q = 1$ case, and so are given by

$$\Omega_R = -U\Omega_l U^\dagger. \tag{2.9}$$

We also have

$$\begin{aligned} \mathrm{d}U &= U\Omega_l = -\Omega_R U \\ \mathrm{d}U^\dagger &= U^\dagger \Omega_R = -\Omega_l U^\dagger. \end{aligned} \tag{2.10}$$

For the Cartan one-form (2.8) we can introduce the linear basis. Following the so-called $4D_+$ bi-covariant calculus of Woronowicz [8, 14] we can choose $\Omega = \omega_A \tau_A$ ($A = 0, 1, 2, 3$), where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & 0 \\ -\frac{1}{q} & 0 \end{pmatrix} \qquad \tau_3 = \frac{1}{1+q^2}\begin{pmatrix} -\frac{1}{q} & 0 \\ 0 & \frac{1}{q} \end{pmatrix} \tag{2.11}$$

and

$$\tau_0 = \frac{q^3 - 1}{(1+q)(1+q^2)} I$$

where $I$ denotes the unit matrix. Observe that we have chosen a non-singular normalisation (compare with [8, second paper, p 108]).

The $\mathrm{SU}_q(2)$ $\sigma$ fields are then introduced by the mapping (1.1) i.e. $U_{ij} \to U_{ij}(x)$. If we now write

$$\begin{aligned} \Omega_{ik} &= U^\dagger_{ij} \frac{\partial U_{jk}}{\partial x_\mu}\,\mathrm{d}x_\mu = \Omega^\mu_{ik}\,\mathrm{d}x_\mu \\ \Omega_\mu &= \omega_{A\mu}\tau_A \end{aligned} \tag{2.12}$$

the action of the $\mathrm{SU}_q(2)$ model can be written as

$$\tilde{S} = -\int \mathrm{d}^d x\, \mathrm{Tr}\,(\Omega_\mu \Omega^\mu) = -\int \mathrm{d}^d x\, G_q^{AB} \omega_{A\mu} \omega_B^\mu \tag{2.13}$$

where $G_q^{AB} = \mathrm{Tr}\,(\tau_A \tau_B)$ is given by

$$G_q^{AB} = \begin{pmatrix} \frac{2(q^3-1)^2}{(1+q)^2(1+q^2)^2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{q} & 0 \\ 0 & -\frac{1}{q} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{q^2(1+q^2)^2} \end{pmatrix}. \tag{2.14}$$

The Cartan forms $\omega_A$ describe the $\sigma$ model currents. We see that the contribution of the scalar current $\omega_0$ vanishes in the limit $q \to 1$. $G_q^{00}$ vanishes at $q = 1$ by the choice of $\tau_0$. This choice is consistent as in this limit we get the SU(2) model and no U(1) current.

We can now consider $\sigma$ fields and take currents $\omega_A$ as our basic variables. Due to the unitarity ($\Omega = U^\dagger \,\mathrm{d}U = -\mathrm{d}U^{\,\dagger} U$) we can rewrite the action (2.13) as

$$\bar{S} = \int \mathrm{d}^d x \, \frac{\partial U^\star_{ji}}{\partial x^\mu} \frac{\partial U_{ji}}{\partial x_\mu} = \int \mathrm{d}^d x \, \frac{\partial (SU)_{ij}}{\partial x^\mu} \frac{\partial U_{ji}}{\partial x_\mu}. \tag{2.15}$$

Denoting

$$\mathrm{U}(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \tag{2.16}$$

we obtain

$$\bar{\mathrm{S}} = \int \mathrm{d}^d x \, (A^\star_{,\mu} A^{,\mu} + q^2 C_{,\mu} C^{\star,\mu} + C^\star_{,\mu} C^{,\mu} + A_{,\mu} A^{\star,\mu} \tag{2.17}$$

where the field operators $A(x), A^\star(x), C(x), C^\star(x)$ satisfy the algebra (2.7) at every coordinate point $x$. But as we have the operators and their derivatives, we need to know the algebra at points $x$ and $x + \epsilon$, with $\epsilon$ infinitesimal. To do this we have to determine the complete algebra for our basic fields, i.e. for $\mathrm{U}_{ij}$ and $\omega_A$. This algebra, in the case of $\mathrm{SU}_q(2)$, is known in an explicit form [5, 14].

### 2.2. *Separable realizations*

Let us assume that

$$U_{ij}(x) = f_{ij}(x) \otimes \hat{U}_{ij} \qquad i, j = 1, 2 \tag{2.18}$$

where the functions $f_{ij}(x)$ are classical and $\hat{U}_{ij}$ describe the coordinate independent operators. Further, let us assume that the quantum $\sigma$ field (2.18) satisfies the unitarity condition $UU^\dagger = U^\dagger U = 1$.

We shall consider here only two cases:

($\alpha$) The operators $\hat{U}_{ij}$ describe the generators of the $\mathrm{SU}_q(2)$ quantum algebra (2.7) separately (obviously the total $U_{ij}$ should).

In this case the unitarity condition, with $f_{11} = f$, $f_{12} = g$, $f_{21} = h$, $f_{22} = k$ and $\hat{c} = \hat{u}_{21}$ is

$$\begin{aligned} &h = g^\star \qquad k = f^\star \\ &|f|^2(1 - q^2 \hat{c}^\dagger \hat{c}) + q^2 |g|^2 \hat{c}^\dagger \hat{c} = 1 \\ &|f|^2(1 - \hat{c}^\dagger \hat{c}) + |g|^2 \hat{c}^\dagger \hat{c} = 1. \end{aligned} \tag{2.19}$$

We see that for $q \neq 1$ we have

$$|f|^2 = |g|^2 = 1 \tag{2.20}$$

i.e. we obtain the $U(1) \times U(1)$ classical $\sigma$ model.

($\beta$) The functions $f_{ij}(x)$ describe the SU(2) $\sigma$ fields and so besides the conditions in the first line of (2.19) they satisfy also

$$|f|^2 + |g|^2 = 1. \tag{2.21}$$

Then, we can write (2.18) as an $SU_q(2)$ matrix

$$U = \begin{pmatrix} f(x) \cdot A & -qg(x) \cdot C^\dagger \\ g(x)^\star \cdot C & f(x)^\star \cdot A^\dagger \end{pmatrix} \tag{2.22}$$

where $A$ and $C$ do not depend on $x$. Thus we obtain for $A$, $A^\dagger$, $C$ and $C^\dagger$ the relations of the first line of (2.7) and

$$\begin{aligned} &|f|^2 AA^\dagger + q^2|g|^2 CC^\dagger = 1 \\ &|f|^2 A^\dagger A + |g|^2 C^\dagger C = 1 \end{aligned} \tag{2.23}$$

whose solutions exist only for $q = 1$, and only when $AA^\dagger = A^\dagger A = CC^\dagger = 1$.

Thus, in conclusion, we see that the assumption (2.18) for $q$ real and with either case $\alpha$) or $\beta$), is too restrictive and should be extended to, say

$$U_{ij}(x) = \sum_n f_{ij}^{(n)}(x) \otimes \hat{U}_{ij}^{(n)} \tag{2.24}$$

where $\hat{U}_{ij}^{(n)}$ describe a polynomial basis of the ring of non-commutative functions $f(G_q)$ (see e.g. [20]). Such an assumption corresponds to the considering of the mapping (1.2) with $S_q = S \otimes G_q$, and $M = GL(2)$.

### 2.3. *Embeddings in the* U($\infty$) *$\sigma$ model*

Another way of representing the operators $A$, $A^\dagger$, $C$ and $C^\dagger$ of (2.17) corresponds to the use of the parameter dependent irreducible representations of the functions $f(SU_q(2))$ in a separable Hilbert space $H$. Promoting the parameters to the functions generates an $\infty$-dimensional $\sigma$ model.

The irreducible representations of the $SU_q(2)$ algebra in a Hilbert space are known [5, 20]. There are only 2 series of irreducible representations of $f(SU_q(2))$, each one parametrised by the parameter of the unit circle $t = e^{i\phi} \in S^2$. One series is degenerate, as for it only the element $a$ of (2.7) is represented in a non-trivial way. The other irreducible representation is non-trivial. It is described by the operators $\rho_\phi$ which act as

$$\begin{aligned} &\rho_\phi(a)e_0 = 0 \qquad \rho_\phi(a)e_k = (1 - q^{-2k})^{\frac{1}{2}} e_{k-1} \\ &\rho_\phi(c)e_k = (e^{i\phi} \cdot q^{-k})e_k \end{aligned} \tag{2.25}$$

where $e_k$, for $k = 0, 1, \ldots \infty$ describes an orthonormal basis in $H$.

If we now replace $\phi$ by a function $\phi(x)$ we obtain from (2.25) the embedding of the U(1) $\sigma$ field into U($\infty$), in analogy with the separable realizations (2.18).

## 3. $U_q(2)$ $\sigma$ model for $q^N = 1$

In order to obtain a solution of a quantum group $\sigma$ model in the separable form (see (2.18)), it will turn out that we should consider the deformation parameter $q$ as being complex and satisfying $|q| = 1$. Then, if $q$ equals the $N$th root of unity ($q^N = 1$) one can represent the operators $\hat{U}_{ij}$ by $N \times N$ dimensional matrices (see [18, 19]). As we want to consider a quantum group $\sigma$ model defined on a quantum compact group, we shall discuss here the simplest such compact quantum group permitting complex values of $q$, namely, the quantum group $U_q(2)$.

### 3.1. *Quantum group $U_q(2)$*

We shall define the quantum group $U_q(2)$, for complex $q$, as a real form of a two parameter deformation of $GL(2, C)$, denoted by $GL_{p,q}(2)$ [21]. Then we will know that the real Hopf algebra is valid for our $U_q(2)$.

The formulae (2.1) have to be extended and they become

$$\begin{aligned} &ab = pba \qquad ac = qca \qquad cd = pdc \\ &pbc = qbc \qquad bd = qdb \qquad ad - da = (p - q^{-1})bc \end{aligned} \tag{3.1}$$

with a coproduct still defined by (2.2). If we now introduce the determinant

$$D = ad - pbc = ad - qcb = da - p^{-1}cb = da - q^{-1}bc \tag{3.2}$$

then one can check from (3.1) that the following relations hold:

$$\begin{aligned} &[D, a] = 0 \qquad [D, d] = 0 \\ &qDb = pbD \qquad pDc = qcD. \end{aligned} \tag{3.3}$$

We see that only if $q = p$ we can put $D = 1$, i.e. we have the quantum group $SL_q(2)$ defined by (2.1).

The quantum group $GL_{p,q}(2)$ is a genuine Hopf algebra for any complex $q$ and $p$. In particular, the antipode of

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.4}$$

is given by the formulae

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = D^{-1}\begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} = \begin{pmatrix} d & -p^{-1}b \\ -pc & a \end{pmatrix} D^{-1} \tag{3.5}$$

where we have used $DD^{-1} = D^{-1}D = 1$. If we now impose the unitarity condition $U^\dagger = S(U)$, i.e.

$$\begin{aligned} a^* &= D^{-1}d = dD^{-1} \\ b^* &= -qD^{-1}c = -pcD^{-1} \\ c^* &= -q^{-1}D^{-1}b = -p^{-1}bD^{-1} \\ d^* &= D^{-1}a = aD^{-1} \end{aligned} \tag{3.6}$$

we find as the consistency conditions, that

$$D^\star = D^{-1} \qquad p = q^\star \tag{3.7}$$

and so obtain the following $U_q(2)$ algebra

$$ac = qca \qquad ac^\star = q^\star c^\star a \qquad c^\star c = cc^\star . \tag{3.8}$$

Moreover, from (3.2) and (3.6) we find that

$$a^\star a + c^\star c = 1 \qquad aa^\star + |q|^2 cc^\star = 1. \tag{3.9}$$

The quantum matrix $U_q(2)$ given by

$$U = \begin{pmatrix} a & -q^\star c^\star D \\ c & a^\star D \end{pmatrix}. \tag{3.10}$$

It describes the generators of the Hopf algebra with standard comultiplication rule

$$\Delta(U_{ik}) = \sum_{j=1,2} U_{ij} \otimes Ujk \tag{3.11}$$

and the following antipode condition (using $qD^{-1}c = q^\star c D^{-1}$)

$$U^\dagger = S(U) = \begin{pmatrix} a^\star & c^\star \\ -qD^{-1}c & D^{-1}a \end{pmatrix}. \tag{3.12}$$

It should be stressed that when $q$ is not real we cannot have $D$ commute with the elements of the matrix, since if it did equations (3.6) and (3.7) would have implied that $q$ is real. Also it should be added that even for $|q| = 1$ but with $q \neq 1$, we cannot put $D = 1$ as in this case (3.9) would give

$$a^\star a + c^\star c = 1 \qquad a^\star a = aa^\star . \tag{3.13}$$

### 3.2. $2N \times 2N$ matrix realization of $U_q(2)$ for $q^N = 1$

If $q^N = 1$ the elements $a$, $c$, $a^\star$ and $c^\star$ can be represented by $N \times N$ matrices. Following [18, 19] we introduce the following matrices†

$$Q = \begin{pmatrix} q & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & q^N \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}. \tag{3.14}$$

These matrices satisfy $PQ = qQP$, and with $q = \exp(2\pi \mathrm{i}/N)$ they generate the algebra of all $N \times N$ matrices. Moreover, as $Q^N = P^N = 1$ we find that $Q^\dagger = Q^{N-1}$ and $P^\dagger = P^{N-1}$ and, as is easy to see,

$$q^{ij} Q^i P^j = P^j Q^i . \tag{3.15}$$

† The matrices (3.13) were introduced earlier by Eguchi and Kawai in their construction of the 'twisted Eguchi–Kawai' models [22].

This observation suggests that if we restrict ourselves to the $q$s being the $N$th root of identity, we can seek a solution of our conditions (3.8) and (3.9) with the elements of $U$ given in terms of various linear combinations of the products of $Q$s and $P$s. The simplest of such solutions corresponds to the case when $a \sim Q$ and $c \sim P$ and so is given by

$$\begin{aligned} a &= \sin\alpha e^{i\psi} P \quad a^\star = \sin\alpha e^{-i\psi} P^{N-1} \\ c &= \cos\alpha e^{i\phi} Q \quad c^\star = \cos\alpha e^{-i\phi} Q^{N-1} \end{aligned} \tag{3.16}$$

where $\alpha$ and $\psi$ are real fields. Then it is easy to check that all conditions (3.7) and (3.8) (with $|q| = 1$) are satisfied if we choose

$$D = e^{i\xi} P^2 \tag{3.17}$$

where $\xi$ is a real field. Hence a $2N \times 2N$ matrix (3.10) with its entries given by (3.16) and (3.17) is a representation of $U_q(2)$ for $q = \exp(2\pi i/N)$.

In the following we shall express the matrix (3.10) as a product of two matrices $U = T\tilde{D}$, $U^\dagger = \tilde{D}^\dagger T^\dagger$, where

$$T = \begin{pmatrix} \sin\alpha e^{i\psi} P & -q^\star \cos\alpha e^{-i\phi} Q^\dagger \\ \cos\alpha e^{i\phi} Q & \sin\alpha e^{-i\psi} P^\dagger \end{pmatrix} \tag{3.18}$$

and

$$\tilde{D} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\xi} P^2 \end{pmatrix}. \tag{3.19}$$

The basic Lagrangian then becomes

$$L = -\tfrac{1}{2} \operatorname{Tr}(U^\dagger \partial_\mu U)(U^\dagger \partial^\mu U) \tag{3.20}$$

with the trace taken with respect to the $U_q(2)$ matrix indices as well as the ones describing the realizations (3.18) and (3.19). One can write

$$\Omega^\mu = U^\dagger \partial^\mu U = \tilde{D}^\dagger L^\mu \tilde{D} + \tilde{D}^\dagger \partial^\mu \tilde{D} \tag{3.21}$$

where

$$L^\mu = T^\dagger \partial^\mu T = \begin{pmatrix} V^\mu & Z^\mu \\ K^\mu & -V^\mu \end{pmatrix}. \tag{3.22}$$

We obtain

$$V^\mu = i(\psi,^\mu \sin^2\alpha + \phi,^\mu \cos^2\alpha) \tag{3.23}$$

and

$$\begin{aligned} Z^\mu &= Q^\dagger P^\dagger y^{\dagger\mu} = Q^\dagger P^\dagger e^{-i\psi - i\phi}[\alpha,^\mu - i(\sin\alpha\cos\alpha)(\psi,^\mu - \phi,^\mu)] \\ K^\mu &= -PQy^\mu = -PQe^{i\psi + i\phi}[\alpha,^\mu + i(\sin\alpha\cos\alpha)(\psi,^\mu - \phi,^\mu)] = -Z^{\mu\dagger} \end{aligned} \tag{3.24}$$

where

$$\rho,^{\mu} = \frac{\partial \rho}{\partial x^{\mu}}. \tag{3.25}$$

Hence $L^{\mu}$ can be resolved into

$$L^{\mu} = V^{\mu}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + y^{\dagger\mu}\begin{pmatrix} 0 & Q^{\dagger}P^{\dagger} \\ 0 & 0 \end{pmatrix} - y^{\mu}\begin{pmatrix} 0 & 0 \\ PQ & 0 \end{pmatrix} \tag{3.26}$$

and so we observe the explicit appearance of the $\mathrm{SU}_q(2)$ algebra generators (which in this case are also the SU(2) generators). Notice that as $D = \mathrm{e}^{\mathrm{i}\xi}P^2$

$$\Omega^{\mu} = \begin{pmatrix} V^{\mu} & Z^{\mu}D \\ D^{-1}K^{\mu} & -\tilde{V}^{\mu} \end{pmatrix} \tag{3.27}$$

where

$$\tilde{V}^{\mu} = V^{\mu} - D^{-1}\partial^{\mu}D = V^{\mu} - \mathrm{i}\partial^{\mu}\xi \tag{2.28}$$

and the Lagrangian (3.20) is given by the formula

$$L = -\tfrac{1}{2}V^{\mu}V_{\mu} + y^{\mu}y^{\dagger}_{\mu} + \tfrac{1}{2}\xi_{,\mu}\xi,^{\mu} - \mathrm{i}V^{\mu}\xi_{,\mu} \tag{3.29}$$

where $\xi(x)$ describes the U(1) field extending SU(2) to U(2). Indeed, one can show that if we put $\xi = 0$ we recover the classical action for the SU(2) $\sigma$ model. On the other hand, we can generalise the Lagrangian (3.20) to

$$L = -\tfrac{1}{2}\,\mathrm{Tr}\,(U^{\dagger}\nabla_{\mu}U)(U^{\dagger}\nabla^{\mu}U) = -\,\mathrm{Tr}\,\tfrac{1}{2}\tilde{\Omega}_{\mu}\tilde{\Omega}^{\mu} \tag{3.30}$$

where we have introduced the U(1) covariant derivative

$$\nabla_{\mu} = \begin{pmatrix} \partial_{\mu}\cdot 1_N & 0 \\ 0 & (\partial_{\mu} - A_{\mu})\cdot 1_N \end{pmatrix} \tag{3.31}$$

and so

$$\tilde{\Omega}^{\mu} = \begin{pmatrix} V^{\mu} & Z^{\mu}D \\ D^{-1}K^{\mu} & -(\tilde{V}^{\mu} + A^{\mu}) \end{pmatrix}. \tag{3.32}$$

Then, if in particular, we choose the pure gauge mode for the $A_{\mu}$ field

$$A_{\mu} = D^{-1}\partial_{\mu}D = \mathrm{i}\xi_{,\mu}\cdot 1_N \tag{3.33}$$

we find that (3.30) reduces to the conventional SU(2) $\sigma$ model. We see that the U(1) gauge field $A_{\mu}$ leads to the appearance of the gauge invariance which allows us to set $D = 1$ in (3.27).

## 4. Outlook

In this paper we have considered some particular solutions of $\sigma$ models taking values in quantum groups. Our main example corresponded to the $U_q(2)$ $\sigma$ model with $q^N = 1$. The 'classical' fields of this model were described by $2N \times 2N$ matrices, i.e. we considered the embedding $U_q(2) \subset U(2N)$. In the general case, however, the embeddings must involve infinite-dimensional $\sigma$ models i.e. $U(\infty)$ or $O(\infty)$. Indeed, the quantum group generators can be represented in terms of the Heisenberg algebra, which can be realised using infinite-dimensional matrices. In fact, it is only when we impose the relation

$$[a, a^\star] = 0 \tag{4.1}$$

that we can realise the generators of $SU_q(2)$ or $U_q(2)$ in terms of finite dimensional matrices. For $SU_q(2)$, equations (2.7) and (4.1) imply $q^2 = 1$. For $U_q(2)$, equations (3.9) and (4.1) imply $|q| = 1$.

We hope that our paper will be treated as a preliminary study of some aspects of sigma models taking values in a quantum group. Let us mention some of these aspects:

(a) For a given quantum group one can consider any finite dimensional representation ($u_{ij} \to T_{AB}(u_{ij})$). For example, for the quantum group $SU_q(2)$ one can consider any $(2j+1)$ - dimensional representation e.g. [23]). If $j = 1$, this would give us the $O_q(3)$ quantum group $\sigma$ model.

(b) The action of the algebraic quantum group $\sigma$ model can be considered as the argument (after exponentiation) of the generalised Feynman path integral provided that the suitable formulae for the integration over the quantum group functions $f(G_q)$ are found. This problem bears some analogy with the Berezin integration rules for Grassmann algebras, and in the case of an arbitrary quantum group, is still to be determined . (See, however, [24] for a discussion of a simple two-dimensional non-commutative case).

(c) The concept of a quantum group $\sigma$ field should be useful when one wants to introduce the notion of generalised gauge fields, with *local* gauge transformations described by quantum group parameters. In section 3 we have introduced the U(1) gauge field but because of its Abelian nature the field's non-commutative aspects were absent. It would be interesting to consider e.g. the $U_q(3)$ $\sigma$ model coupled to non-Abelian $U_q(2)$ gauge fields. This should allow us (by gauge fixing) to formulate the quantum group $\sigma$ model on the coset $U_q(3)/U_q(2)$.

Finally, we would like to add that although in this paper we have considered $\sigma$ fields taking values in non-commutative algebras the two best known choices, namely, the quaternionic algebra (see e.g. [25]) and the Grassmann algebra (see e.g. [26]) are finite dimensional. In the case of quantum groups, for a generic $q$, the algebra of non-commuting functions $f(G_q)$ is infinite dimensional and so in order that we can extend e.g. the superfield formalism of supersymmetric theories to the case of quantum groups, the new formal tools still have to be developed.

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